The Competence Description in Micro 3 says:

Game Theory has become a central analytic tool in much economic theory, e.g. within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

The course aims at giving the student knowledge of game theory, non-cooperative as well as cooperative, and its applications in economic models.

The student who successfully completed the course will learn the basic game theory and will be enabled to work further with advanced game theory. The student will also learn how economic problems, involving strategic situations, can be modeled using game theory, as well as how these models are solved. The course intention is thus, that the student through this becomes able to work with modern economic theory, for instance within the areas of within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

In the process of the course the student will learn about

- *Static games with complete information*
- *Static games with incomplete information*
- *Dynamic games with complete information*
- *Dynamic games with incomplete information*
- *Basic cooperative game theory.*

For each of these classes of games, the student should know and understand the theory, and learn how to model and analyze some important economic issues within the respective game framework.

More specifically, the students should know the theory and be able to work with both normal and extensive form games. They should know, understand and be able to apply the concepts of dominant strategies, iterative elimination of dominant strategies, as well as mixed strategies. The students should know the central equilibrium concepts in non-cooperative game theory, such as Nash Equilibrium and further refinements: Subgame-Perfect Nash Equilibrium, Bayesian Nash Equilibrium, Perfect Bayesian Equilibrium. They should understand why these concepts are central and when they are used, and be able to apply the relevant equilibrium and solution concepts.

Furthermore, the students should acquire knowledge about a number of special games and the particular issues associated with them, such as repeated games (including infinitely repeated games), auctions and signaling games.

The students should also understand and be able to apply the solution concepts of cooperative game theory, such as the core. Furthermore, the students should also learn the basics of bargaining theory.

To obtain a top mark in the course the student must be able excel in all of the areas listed above.

In view of this, the grading of the exam should take as a point of departure, the short description of the solutions below

MICRO 3 EXAM August 2010
QUESTIONS WITH SHORT ANSWERS
(The answers in this solution can often be short/indicative, a good exercise should argue for these answers)

1. (a) Find *all* Nash equilibria in the following game

	L	R
T	$2,3$	$0,2$
B	$1,0$	$4,7$

Solution: The are two PSNE, (T,L) and (B,R). The mixed eq can be determined as follows: assume that all pure strategies are played with non-negative probability and assign p as the probability that player 1 plays T and q as the probability that player 2 plays L.

		q	1-q
		L	R
r	T	$2,3$	$0,2$
1-r	B	$1,0$	$4,7$

Row player is indifferent between playing T and B iff

$$\begin{aligned} 2q + 0*(1-q) &= q + 4(1-q) \Leftrightarrow \\ q &= 4/5. \end{aligned}$$

Row player's best response is

$$BR_1(q) = r^*(q) \begin{cases} = 1 \text{ if } q > 4/5 \text{ (strategy T)} \\ \in [0,1] \text{ if } q = 4/5 \text{ (any combination of } T \text{ and } B) \\ = 0 \text{ if } q < 4/5 \text{ (strategy B)} \end{cases}$$

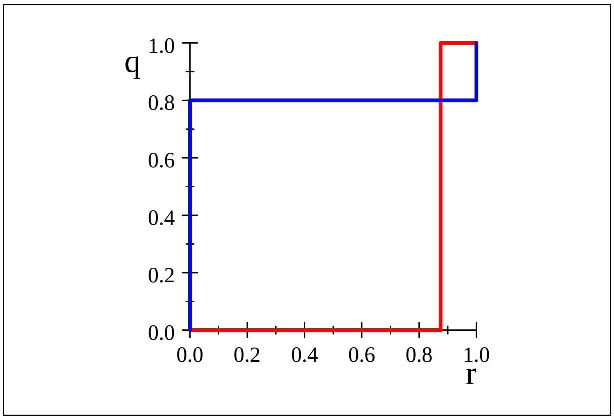
Column player is indifferent between playing L and R iff

$$3r + 0*(1-r) = 2r + 7(1-r),$$

that is, if the row player is mixing with the weight $r = 7/8$. Column player's best response is

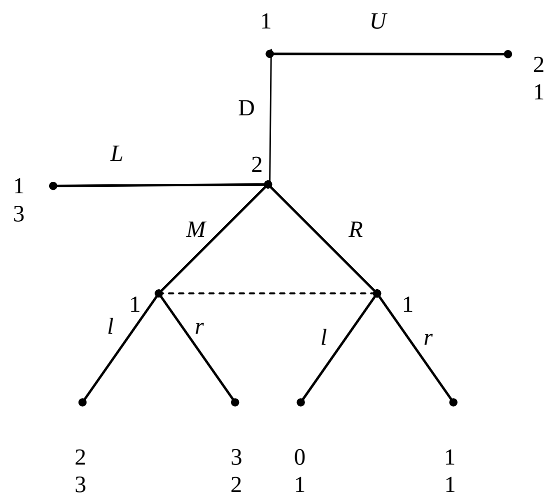
$$BR_2(r) = q^*(r) \begin{cases} = 1 \text{ if } r > 7/8 \text{ (strategy L)} \\ \in [0,1] \text{ if } r = 7/8 \text{ (any combination of } L \text{ and } R) \\ = 0 \text{ if } r < 7/8 \text{ (strategy R)} \end{cases}$$

The intersection of BRs is (the BR of Player 1 is in blue, and the BR of player 2 is in red)



Therefore, the mixed strategy equilibrium is $[(7/8, 1/8)(4/5, 1/5)]$, i.e. the row player plays T with prob 7/8, and the column player plays L with prob 4/5.

(b) Consider the extensive-form game below



i. Describe the strategy sets of all the players and count the number of subgames.
Solution: There are two players in this game. Player 1 has 4 strategies, Ull, Urr, Dll, and Drr. Player 2 has 3 strategies, L, M and R. There are two subgames including the game itself.

ii. Find all its (pure-strategy) subgame perfect Nash equilibria.
Solution: Consider first the subgame that starts in the node controlled by Player 2. In the normal form it would look like

		Player 2		
		L	M	R
Player 1	ll	$\underline{1},\underline{3}$	$2,\underline{3}$	$0,1$
	rr	$\underline{1},\underline{3}$	$\underline{3},2$	$\underline{1},1$

It has two (pure strategy) Nash equilibria - (ll, L) and (rr, L). Now move backward to the entire game. Given the two NE of the subgame, Player 1 will choose U (with the payoff of 2), as opposed to D (which would pay 1 no matter if (ll, L) or (rr, L) is played in the continuation game). Therefore, there are two SPNE in this extensive form game: $(Ull, L)$and (Urr, L).

(c) Consider the following game between two players

	C	D
C	$2, 2$	$-1, 3$
D	$3, -1$	$1, 1$

i. Solve this game by iterated elimination of strictly dominated strategies
Solution: For both players C is strictly dominated by D. Solution is (D,D).

ii. Assume that this game is repeated infinitely many times $t = 1, 2, 3, ..., \infty$, and both players maximize the sum of their own future discounted payoffs. Assume further that both players have a common discount factor $\delta = 2/3$. Propose a "grim trigger strategy" SPNE of this game, such that outcome (C,C) is played in each period *along the game path.* Demonstrate that your proposal is indeed supported as an SPNE (i.e. that no player wants to deviate from the proposed rule).
Solution: Consider the following "grim trigger strategy" profile for both players:

- In the first period play C. If in all past periods both players played C, play C (normal phase)
- If either of players played D at any period in the past, play D from now on (punishment phase).

Let's show that this strategy profile constitutes a SPNE of this infinitely repeated game. Assume one of the players (say, Player 2), sticks to the strategy above, and show that Player 1 will not deviate if $\delta = 2/3$. Start from the normal phase subgames. If Player 1 sticks to the strategy above, she gets

$$2 + \delta * 2 + \delta^2 * 2 + ... = \frac{2}{1-\delta} = \frac{2}{1-2/3} = 6.$$

If she deviates, than the highest payoff that she can get in the first period is the payoff that she obtains when she plays her best response to Player 2 playing C. This best response (=best deviation) is D and her deviation payoff is then 3. However, from period 2 onwards the game would enter punishment phase, in which Player 1 will get only $U_1(D, D) = 1$ each period. Therefore her payoff from deviation is

$$3 + \delta * 1 + \delta^2 * 1 + ... = 3 + \frac{\delta}{1-\delta} = 3 + \frac{2/3}{1-2/3} = 5.$$

So, Player 1 does not deviate in normal phase as $6 > 5$.Notice, that Player 1 does not have an incentive to deviate in the punishment phase either, as in that phase Player 2 plays the stage game NE strategy *no matter what Player 1 does*, so the best response of Player 1 is to play the stage game NE (D,D) as well.
Similar logic holds for Player 1. So, the proposal above is a SPNE.

iii. Consider the following statement: "Assume that a player has a *strictly dominant* strategy s in a stage game G (i. e. the strategy that makes him better off than any of his other strategy, no matter what the other players do). Now assume that G is repeated infinitely many times, and players maximize the sum of their future discounted payoffs. Then playing s in each period of $G(\infty)$ would also give this player the highest payoff". True or false? Explain your answer.
Solution: False. As an example, see above (D is strictly dominant for either player, but playing (D,D) in each period is worse than playing the grim trigger strategy).

2. Two neighboring countries are contributing to the monitoring of the activity of a nearby volcano. They simultaneously and non-cooperatively decide on level of the technology they want to use at their respective geological stations. If country 1 chooses level $x_1 \in [0, 1]$ and country 2 chooses level $x \in [0, 1]$, then the probability that they will detect an upcoming volcano eruption is

$$\frac{(x_1 + 1)(x_2 + 1)}{4}.$$

The cost of obtaining a technology level x_1 in country 1 is $\frac{(x_1)^2}{4}$, and is known to both countries. The cost of obtaining a technology level x_2 in country 2 is $\frac{t(x_2)^2}{4}$, where $t > 0$. Each country maximizes the probability of detecting the eruption less the technology cost. That is, the payoff to country 1 is

$$u_1(x_1, x_2) = \frac{(x_1 + 1)(x_2 + 1)}{4} - \frac{(x_1)^2}{4},$$

and the payoff to country 2 is

$$u_2(x_1, x_2) = \frac{(x_1 + 1)(x_2 + 1)}{4} - \frac{t(x_2)^2}{4}.$$

(a) Assume that $t = 1$, and it is known to both countries. Derive the best response functions of both countries and find the technology levels chosen by the countries in NE of this game.
Solution: Country 1 solves

$$\max_{x_1} \frac{(x_1 + 1)(x_2 + 1)}{4} - \frac{(x_1)^2}{4}.$$

The FOC yields

$$\frac{(x_2+1)}{4} - \frac{2x_1}{4} = 0,$$

so the BR of country 1 is

$$x_1 = \frac{x_2+1}{2}. \tag{1}$$

Similarly, country 2 solves

$$\max_{x_2} \frac{(x_1+1)(x_2+1)}{4} - \frac{t(x_2)^2}{4},$$

The FOC yields

$$\frac{(x_1+1)}{4} - \frac{2tx_2}{4} = 0,$$

so the BR of country 2 is

$$x_2 = \frac{x_1+1}{2t}. \tag{2}$$

For t=1 the system of BRs of both countries is

$$\begin{cases} x_1 = \frac{x_2+1}{2} \\ x_2 = \frac{x_1+1}{2} \end{cases},$$

so in the NE the countries choose

$$x_1 = x_2 = 1.$$

(b) Assume instead that $t = 2$, i.e. that the technology in country 2 is more costly. Again, both countries know that $t = 2$. Derive the best response functions of both countries and find the technology levels chosen by the countries in NE. Is country 1 choosing a higher or a lower level of technology in (b), as compared to (a)? Why? Provide some intuition.

Solution: Applying $t = 2$ to the best responses (1) and (2) above we get the following system to determine the equilibrium level of technologies

$$\begin{cases} x_1 = \frac{x_2+1}{2} \\ x_2 = \frac{x_1+1}{4} \end{cases},$$

Solving this system we get

$$\begin{cases} x_1 = \frac{\frac{x_1+1}{4}+1}{2} \\ x_2 = \frac{x_1+1}{4} \end{cases} \Leftrightarrow$$
$$\begin{cases} 8x_1 = x_1 + 1 + 4 \\ x_2 = \frac{x_1+1}{4} \end{cases} \Leftrightarrow$$
$$\begin{cases} x_1 = \frac{5}{7} \\ x_2 = \frac{3}{7} \end{cases}.$$

Both country 1 and country 2 are choosing a lower level of technology than in (a). The intuition is that technology is now more costly for country 2, so it decides to invest less in it. As a result, a extra unit of technology of country 1 has now less influence on the probability of eruption detection (as the latter is given by $\frac{(x_1+1)(x_2+1)}{4}$). Thus, country 1 decides to invest less in its technology as well. In other words, country 1's effort to get a better technology now pays worse than in (a), which makes country 1 choose a lower technology level.

(c) Now assume that t, the exact value of the technology cost parameter of country 2, is only known to country 2. Country 1 only knows that t can be either $t_L = 1$ with probability 1/3, or $t_H = 2$ with probability 2/3.

i. What are the types of both players in this static game of incomplete information? What are the strategies of the players?
Solution: Country 1 only has 1 type, country 2 has 2 types - high cost (H) and low cost (L). Strategy of country 1 is technology level x_1, strategy of country 2 is a pair of technology levels for both types (x_2^L, x_2^H).

ii. What is the best response of country 2 with the technology cost parameter $t_L = 1$, $x_2^L(x_1)$? Of country 2 with $t_H = 2$, $x_2^L(x_1)$?
Solution: Country 2 has complete information in this case. As a result, its BRs are given by the expression (2) for $t_L = 1$ or $t_H = 2$ respectively, i.e.

$$x_2^L(x_1) = \frac{x_1+1}{2} \tag{3}$$

$$x_2^H(x_1) = \frac{x_1+1}{4} \tag{4}$$

iii. What is the best response of country 1, $x_1(x_2^L, x_2^H)$?
Solution: Country 1 is uncertain about the cost of country 2. It solves

$$\max_{x_1} \frac{1}{3}\left[\frac{(x_1+1)\left(x_2^L+1\right)}{4} - \frac{(x_1)^2}{4}\right] + \frac{2}{3}\left[\frac{(x_1+1)\left(x_2^H+1\right)}{4} - \frac{(x_1)^2}{4}\right]$$

$$= \max_{x_1} \frac{1}{3}\frac{(x_1+1)\left(x_2^L+1\right)}{4} + \frac{2}{3}\frac{(x_1+1)\left(x_2^H+1\right)}{4} - \frac{(x_1)^2}{4}$$

The FOC is

$$\frac{1}{3}\frac{\left(x_2^L+1\right)}{4} + \frac{2}{3}\frac{\left(x_2^H+1\right)}{4} - \frac{2x_1}{4} = 0,$$

so the BR of country 1 in this case is

$$x_1(x_2^L, x_2^H) = \frac{1}{2}\left(\frac{x_2^L + 2x_2^H}{3} + 1\right) \tag{5}$$

iv. Find the Bayes-Nash equilibrium of this game.
Solution: Solving the system of BRs (3), (4) and (5), we get

$$\begin{cases} x_2^L = \frac{x_1+1}{2} \\ x_2^H = \frac{x_1+1}{4} \\ x_1 = \frac{1}{2}\left(\frac{x_2^L+2x_2^H}{3}+1\right) \end{cases} \Leftrightarrow \begin{cases} x_2^L = \frac{x_1+1}{2} \\ x_2^H = \frac{x_1+1}{4} \\ x_1 = \frac{1}{2}\left(\frac{1}{3}(x_1+1)+1\right) \end{cases} \Leftrightarrow \begin{cases} x_2^L = \frac{x_1+1}{2} \\ x_2^H = \frac{x_1+1}{4} \\ 6x_1 = (x_1+1)+3 \end{cases} \Leftrightarrow$$

$$\begin{cases} x_2^L = \frac{9}{10} \\ x_2^H = \frac{9}{20} \\ x_1 = \frac{4}{5} \end{cases}.$$

In BNE $x_1 = \frac{4}{5}$, $x_2^L = \frac{9}{10}$ and $x_2^H = \frac{9}{20}$.

3. Anna and Bjorn are playing the alternating-offer bargaining game in order to divide 1 unit of (infinitely divisible) output. Their preferences are given just by the amount of output they get. Anna discounts future at the rate $\delta_A \in (0,1)$, and Bjorn discounts future at the rate $\delta_B \in (0,1)$, such that .

   (a) Assume that the game lasts for 3 periods. In period 1 Anna makes an offer on how to split the output: s_1 to herself, $1 - s_1$ to Bjorn. Bjorn observes the offer and decides whether to accept or reject it. If the offer is accepted, the game is over and the players get the accepted offers. Otherwise the game proceeds to period 2. In period 2 Bjorn makes an offer: s_2 to Anna and $1 - s_2$ to Bjorn himself. Anna observes the offer and chooses to accept or to reject it. Again, if the offer is accepted, the game is over, otherwise it proceeds to period 3.

In period 3 Anna receives (exogenously determined) share s of output and Bjorn receives share $1-s$.

What are the payoffs that Anna and Bjorn get in a SPNE of this game? (you may assume that whenever a player is indifferent between accepting and rejecting the offer, she or he accepts.)

Solution: Let's solve this by backward induction. In period 3 there are no active moves by the players, just realization of payoffs. In the second part of period 2 Anna would say "yes" to all offers that bring her at least the payoff she would get in period 3 discounted to period 2's value $\delta_A s$.Bjorn will offer Anna just that

$$s_2 = \delta_A s$$

in order to get the offer accepted, which will leave him with

$$1 - s_2 = 1 - \delta_A s.$$

Notice that Bjorn would prefer getting his offer accepted as long as

$$1 - \delta_A s \geq \delta_B (1 - s),$$

which holds as $\delta_B \geq \delta_A$.

Now turn to second part of period 1. Bjorn will accept any offers that as at least as good as what he gets by not accepting them and proceeding to 2nd period, i.e., $\delta_B (1 - \delta_A s)$. Anna will offer him just that

$$1 - s_1 = \delta_B (1 - \delta_A s),$$

which leaves her with

$$s_1 = 1 - \delta_B (1 - \delta_A s).$$

Again, notice that if instead she makes such an offer that does not get accepted, she gets $\delta_A s_2 = \delta_A (\delta_A s)$, and this is below s_1

$$s_1 - \delta_A s_2 = 1 - \delta_B (1 - \delta_A s) - \delta_A (\delta_A s) = 1 - \delta_B + \delta_A s (\delta_B - \delta_A) > 0.$$

So, in SPNE Anna offers the following split-up:

$$\begin{aligned} s_1 &= 1 - \delta_B (1 - \delta_A s) \text{ to herself,} \\ 1 - s_1 &= \delta_B (1 - \delta_A s) \text{ to Bjorn,} \end{aligned}$$

Bjorn accepts the offer immediately and the game ends in the 1st period.

(b) Now assume that the alternating-offer bargaining game between Anna and Bjorn runs for potentially infinite number of periods. Again, Anna is the one who makes the first offer. Show that the payoff of Anna in the SPNE of this game is given by

$$s_A = \frac{1 - \delta_B}{1 - \delta_A \delta_B},$$

and the payoff of Bjorn is given by

$$s_B = \frac{\delta_B (1 - \delta_A)}{1 - \delta_A \delta_B}.$$

(HINT: we have argued in the lectures that for the infinite-period alternating offer game the subgame that starts in period 3 looks exactly like the subgame that starts in period 1. What does it imply for the payoff that Anna receives in the game that starts in period 3, and in the game that starts in period 1?).

How does the payoff of Anna depend on her own discount factor? Of discount factor of Bjorn? Provide intuition behind your answer.

Solution: As shown above, if Anna receives the payoff of s in the game that starts in period 3, she will receive the payoff of $1 - \delta_B (1 - \delta_A s)$ in the game that starts in period 1. As the games that start in period 1 and in period 3 are identical, so should be the payoffs in these games. Using this logic, we can equalize these payoffs

$$s = 1 - \delta_B (1 - \delta_A s)$$

and solve this equation for s, which yields the payoff of Anna given by

$$s (1 - \delta_A \delta_B) = 1 - \delta_B$$
$$s_A = s = \frac{1 - \delta_B}{1 - \delta_A \delta_B},$$

Bjorn gets

$$s_B = 1 - s_A = \frac{\delta_B (1 - \delta_A)}{1 - \delta_A \delta_B}.$$

Consider the payoff of Anna, s_A.

$$\begin{aligned} \frac{\partial s_A}{\partial \delta_A} &= \delta_B \frac{1 - \delta_B}{(1 - \delta_A \delta_B)^2} > 0, \\ \frac{\partial s_A}{\partial \delta_B} &= \frac{-(1 - \delta_A \delta_B) - (1 - \delta_B)(-\delta_A)}{(1 - \delta_A \delta_B)^2} = \frac{-(1 - \delta_A)}{(1 - \delta_A \delta_B)^2} < 0 \end{aligned}$$

That is, the more patient is Anna, the less if she afraid of postponing the agreement to the next period and the higher is her payoff. Similarly, the more patient is Bjorn, the less is he willing to accept whatever is offered now, as opposed to waiting till tomorrow, which makes Anna to "keep" less for herself in her offer.